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BRST algebra on quantum bundles

H.-D. Doebner [a], M. Tahiri [a,b,*]

[a] *Arnold Sommerfeld Institut für Mathematische Physik, Technische Universität Clausthal, 38678 Clausthal-Zellerfeld, Germany*

[b] *Laboratoire de Physique Théorique, Université d'Oran Essenia, 31100 Oran, Algeria*

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Abstract

A quantum analogue of the BRST algebra is given. The method is based on the construction of a differential algebra of generalized quantum forms carrying a bigraduation. This is realized over the base quantum space of a quantum vector bundle associated to a quantum principal bundle. Using this approach, we introduce the quantum gauge, ghost and matter fields via connections and sections. Imposing constraints on the curvatures leads to the quantum BRST transformations of these fields. © 2000 Elsevier Science B.V. All rights reserved.

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Gauge theories of quantum groups have been investigated recently in many papers (see [6]; and references therein). The main goal of these papers is to construct gauge theories in the realm of noncommutative geometry by taking quantum groups as gauge groups.

Moreover, in the quantization procedure of usual gauge theories, one needs, besides the classical fields, the ghost fields, and the corresponding quantized gauge theories are invariant under the BRST transformations [1,8]. It is tempting to try and construct a quantum gauge theory reflecting the fields and their BRST transformations present in the standard quantized gauge theory. There has been a proposal to this problem [9]. Here, the gauge, ghost and matter fields as well as the corresponding BRST transformations are introduced using the bicovariant differential calculus on the quantum group $SU_q(2)$.

In this paper we will approach this problem by an algebraic formulation as developed in [3]. The basic idea is to perform a straightforward dualization of the fibre bundle structure of ordinary gauge theories. Let us recall that ordinary gauge theories square naturally with fibre

* Corresponding author.

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bundle theory. The gauge fields are represented by a connection on a locally trivial principal bundle with the space–time as base and the gauge group as structure group, whereas the matter fields are represented by sections on an associated vector bundle. The local gauge transformations of the gauge and matter fields are introduced via the local trivializations. To recast these notions in the context of noncommutative differential geometry, we shall use quantum (noncommutative) bundles. Here, the structure group (gauge group) becomes a quantum group which is a noncommutative Hopf algebra $(A, \Delta, \varepsilon, S)$, where Δ is the comultiplication, ε the counit, and S the antipode. The base becomes a quantum space–time which is a noncommutative unital algebra B. The fibre becomes a left A-comodule unital algebra (V, Δ_{L}), where $\Delta_{\mathrm{L}} : V \to A \otimes V$ is the left coaction of A on V (see [3], and references therein).

Furthermore, in order to define the matter, gauge and ghost fields and to find the corresponding BRST transformations in this algebraic setting, we proceed as follows. First, we recall that the fields X_n^g occurring in the quantization of classical gauge theories can be indexed by two grades (n, g), where n represents the degree of the field as an n-form and g its ghost number. However, the exterior differential d and the BRST operator Q are derivations in the graded sense (the grading is $\mathrm{gr}(X_n^g) = n + g \bmod 2$) acting on the space of fields X_n^g such that dX_n^g is of type $(n+1, g)$ and QX_n^g is of type $(n, g+1)$. They satisfy $d^2 = Q^2 = dQ + Qd = 0$.

To translate this description at the algebraic level, we use a locally trivial quantum principal bundle $P(B, A)$ and its quantum associated vector bundle $E(B, V, A)$. We also need to construct over the base quantum space B a graded differential algebra whose elements carry a bigraduation. To do this, we introduce two bimodules $\Gamma^{(1,0)}$ and $\Gamma^{(0,1)}$ over the base B, and we define a first-order differential calculus $(\Gamma^1, \tilde{d})$ over B, where the bimodule Γ^1 is given by

$$\Gamma^1 = \Gamma^{(1,0)} \oplus \Gamma^{(0,1)}. \tag{1}$$

$\tilde{d} : B \to \Gamma^1$ is a linear map satisfying the Leibniz rule

$$\tilde{d}(ab) = (\tilde{d}a)b + a(\tilde{d}b), \quad a, b \in B, \tag{2}$$

and any element $\rho \in \Gamma^1$ is of the form

$$\rho = \sum_k a_k \tilde{d} b_k, \quad a_k, b_k \in B. \tag{3}$$

Moreover, let $(\Omega(B) = \sum_{n=0} \Omega^n(B), \tilde{d})$ be the graded differential algebra built over $(\Gamma^1, \tilde{d})$ [4]. Here $\Omega^n(B)$ for $n > 0$ $(\Omega^0(B) = B)$ is defined as a set spanned by the elements

$$\omega_n = (a_0, a_1, \ldots, a_n) = a_0 \otimes \tilde{d}a_1 \otimes \cdots \otimes \tilde{d}a_n \tag{4}$$

for any $a_0, a_1, \ldots, a_n \in B$, where $\otimes$ is the tensor product over the algebra B. We will omit $\otimes$ in all formulas below. The product of $\omega_n = (a_0, a_1, \ldots, a_n) \in \Omega^n(B)$ and

$\Omega_m = (b_0, b_1, \dots, b_m) \in \Omega^m(B)$ is defined by

$$\omega_n\omega_m = (a_0, a_1, \dots, a_{n-1}, a_n, b_0, b_1, \dots, b_m) + \sum_{i=0}^{n-1}(-1)^{n-i}(a_0, a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_n, b_0, b_1, \dots, b_m). \quad (5)$$

This relation is associated to the right property of $\Omega(B)$ obtained by using (2) to pull the elements of B through the left.

The action of $\tilde{d}$ on $\Omega(B)$ is defined by

$$\tilde{d}(a_0, a_1, \dots, a_n) = (1_B, a_0, a_1, \dots, a_n), \quad (6)$$

$$\tilde{d}(1_B, a_0, a_1, \dots, a_n) = 0. \quad (7)$$

These relations are equivalent to the requirements

$$\tilde{d}1_B = 0, \qquad \tilde{d}^2 = 0. \quad (8)$$

However, the linear operator $\tilde{d}$ satisfies the graded Leibniz rule with respect to the product given by (5). The grading is introduced by $\mathrm{gr}(\omega_n) = n \bmod 2$.

What is important to see in this construction is that in view of (1) and (3), we can write $\tilde{d}$ acting on B as

$$\tilde{d} = d + Q, \quad (9)$$

where $d : B \to \Gamma^{(1,0)}$, $Q : B \to \Gamma^{(0,1)}$ are linear maps satisfying the Leibniz rule so that (2) is also verified.

Therefore, we observe that the operators d and Q acting on B permit us to define a bigrading for the elements of $\Gamma^1 = \Gamma^{(1,0)} \oplus \Gamma^{(0,1)}$. We will say that the elements $\Gamma^{(1,0)}$ and $\Gamma^{(0,1)}$ are of type (1,0) and (0,1), respectively. This can be used to put each part $\Omega^n(B)$ of the differential algebra $\Omega(B)$ in the form

$$\Omega^n(B) = \sum_{g=0}^{n} \Omega^{(n-g,g)}(B), \quad n > 0. \quad (10)$$

Indeed, inserting (9) into (4), we obtain

$$\omega_n = \sum_{g=0}^{n} \omega_{(n-g,g)} = \sum_{g=0}^{n} (a_0, a_1, \dots, a_n)_{(n-g,g)}, \quad (11)$$

where each term $\omega_{(n-g,g)} = (a_0, a_1, \dots, a_n)_{(n-g,g)}$ is given by

$$a_0 da_1 \cdots da_{n-g} Qa_{n-g+1} \cdots Qa_n$$

plus all other terms obtained by permutation of the operators d and Q. Thus we get the decomposition (10), where $\Omega^{(n-g,g)}(B)$ is spanned by the elements $\omega_{(n-g,g)}$ characterized by two grades. For example,

$$\omega_1 = (a_0, a_1) = a_0 da_1 + a_0 Qa_1 = \omega_{(1,0)} + \omega_{(0,1)} \in \Omega^1(B) \equiv \Gamma^1,$$

$$\omega_2 = (a_0, a_1, a_2) = a_0 da_1 da_2 + (a_0 da_1 Qa_2 + a_0 Qa_1 da_2) + a_0 Qa_1 Qa_2$$
$$= \omega_{(2,0)} + \omega_{(1,1)} + \omega_{(0,2)} \in \Omega^2(B) = \Omega^{(2,0)}(B) \oplus \Omega^{(1,1)} \oplus \Omega^{(0,2)}(B).$$

Furthermore, let us define the product similar to that in (5) of $\omega_{(n-g,g)} = (a_0, a_1, \dots, a_n)_{(n-g,g)} \in \Omega^{(n-g,g)}(B)$ and $\omega_{(m-g',g')} = (b_0, b_1, \dots, b_m)_{(m-g',g')} \in \Omega^{(m-g',g')}(B)$ by

$$\omega_{(n-g,g)}\omega_{(m-g',g')} = (a_0, a_1 \dots, a_{n-1}, a_n, b_0, b_1, \dots, b_m)_{(n+m-G,G)} + \sum_{i=0}^{n-1} (-1)^{n-i} (a_0, a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_n, b_0, b_1, \dots, b_m)_{(n+m-G,G)}, \tag{12}$$

where $G = g + g'$. We have $0 \le G \le n + m$, since $0 \le g \le n$ and $0 \le g' \le m$. Notice the consistency of (12) with the product given by (5). In fact, a simple calculation leads to

$$\omega_n \omega_m = \sum_{g=0}^{n} \sum_{g'=0}^{m} N_{n,m,G}\, \omega_{(n-g,g)}\omega_{(m-g',g')} \tag{13}$$

with

$$N_{n,m,G} = \begin{cases} 1/(G+1), & 0 \le G \le \min(n,m) - 1, \\ 1/(\min(n,m)+1), & \min(n,m) \le G \le \max(n,m), \\ N_{n,m,n+m-G}, & \max(n,m) + 1 \le G \le n+m. \end{cases}$$

Now we will extend the action of the linear operators d and Q to the whole of $\Omega(B)$, so that the consistency with (9) is guaranteed. To this end, let us define the action of d on $\Omega^{(n-g,g)}(B)$ $(n > 0, 0 \le g \le n)$ by

$$d(a_0, a_1, \dots, a_n)_{(n,0)} = (1_B, a_0, a_1, \dots, a_n)_{(n+1,0)},$$
$$d(a_0, a_1, \dots, a_n)_{(n-g,g)} = \tfrac{1}{2}(1_B, a_0, a_1, \dots, a_n)_{(n+1-g,g)}, \quad 1 \le g \le n. \tag{14}$$

Similarly we introduce the action of Q by

$$Q(a_0, a_1, \dots, a_n)_{(n-g,g)} = \tfrac{1}{2}(1_B, a_0, a_1, \dots, a_n)_{(n-g,g+1)}, \quad 0 \le g \le n-1,$$
$$Q(a_0, a_1, \dots, a_n)_{(0,n)} = (1_B, a_0, a_1, \dots, a_n)_{(0,n+1)}. \tag{15}$$

According to (6) and (11), it is now easy to see that

$$\tilde{d}(a_0, a_1, \dots, a_n) = (1_B, a_0, a_1, \dots, a_n) = \sum_{g=0}^{n+1} (1_B, a_0, a_1, \dots, a_n)_{(n+1-g,g)}$$
$$= (d+Q) \sum_{g=0}^{n} (a_0, a_1, \dots, a_n)_{(n-g,g)}$$
$$= (d+Q)(a_0, a_1, \dots, a_n). \tag{16}$$

However in view of $\tilde{d}1_B = 0$, we have

$$d1_B = 0, \qquad Q1_B = 0. \tag{17}$$

Therefore, using (14) and (15), we get

$$d^2 = 0, \qquad Q^2 = 0, \tag{18}$$

and according to $\tilde{d}^2 = (d + Q)^2 = 0$, we obtain

$$dQ + Qd = 0. \tag{19}$$

On the other hand, the linear operators d and Q acting on the spaces $\Omega^{(n-g,g)}(B)$ satisfy the graded Leibniz rule with respect to the product given by (12). The proof consists of a straightforward verification by using (12), (14) and (15). The grading is given by $\mathrm{gr}(\omega_{(n-g,g)}) = n \bmod 2$, so that the consistency with the graded Leibniz rule of $\tilde{d} = d + Q$ is also guaranteed by using (13).

So, we have constructed a differential algebra $\Omega(B)$ over the base quantum space B, and the decomposition (10) permits us to see that the elements $\omega_{(n-g,g)}$ of $\Omega(B)$ are supplied with two grades. The graded derivative $d(Q)$ increases the first (second) grade by one. This is a direct consequence of (14) and (15). They satisfy the relations (18) and (19). Thus, we have obtained a construction which has a resemblance with what happens in the usual BRST formalism. Hereafter, we will call the elements $\omega_{(n-g,g)}$ as generalized quantum differential forms of type $(n - g, g)$, where the first grade is the degree of this form and the second one its ghost number. The graded derivatives d and Q play the role of the differential and the BRST operator, respectively.

However, in order to reproduce the fields and their BRST transformations in this algebraic formulation we need the notion of a connection on the quantum principal bundle $P(B, A)$ [3]. The latter is locally trivial, i.e. locally, it looks like $P = A \otimes B$, but globally, the fibre bundle can be twisted through the local trivializations. Similarly, the quantum vector bundle $E(B, V, A)$ associated to $P(B, A)$ locally looks like $E = V \otimes B$. As shown in [3] and working in the local picture, from any local trivialization there arises a local gauge transformation defined as a convolution invertible map $\gamma : A \to B$ with $\gamma(1_A) = 1_B$, so that a section of $E(B, V, A)$ defined as a map $\sigma : V \to B$ transforms under the action of γ as $\sigma \to \gamma * \sigma$, where $*$ is the convolution product. Denoting by $S^n(E)$ the set of maps $V \to \Omega^n(B)$, where $\Omega(B) = \sum_{n=0}\Omega^n(B)$ is the graded differential algebra associated to some first-order differential calculus (Γ, δ) over B, one defines the covariant derivative as a linear map $D : S^0(E) \to S^1(E)$ such that $D\sigma$ transforms under γ as $D\sigma \to \gamma * D\sigma$. Notice that $S^0(E)$ represents the set of sections, since $\Omega^0(B) = B$. It has been established that if a map $\beta : A \to \Gamma$ transforms under γ as $\beta \to \gamma * \beta * \gamma^{-1} + \gamma * \delta\gamma^{-1}$ then the covariant derivative is given by $D = \delta + \beta *$. The map β is called the connection on $P(B, A)$. Moreover, to any connection β one can associate its curvature $F : A \to \Omega^2(B)$ defined as $F = \delta\beta + \beta * \beta$ and satisfying the Bianchi identity $\delta F + \beta * F - F * \beta = 0$. For clarity, we recall some facts about the convolution product $*$. If $\gamma_1, \gamma_2 : A \to B$ are two linear maps, then $\gamma_1 * \gamma_2 : A \to B$ is defined as $\gamma_1 * \gamma_2 = m(\gamma_1 \otimes \gamma_2)\Delta$, where m is the product in B and Δ the comultiplication in A. Then we have $\gamma_1 * \gamma_2(a) = \sum_k \gamma_1(a_k)\gamma_2(b_k)$, since $\Delta(a) = \sum_k a_k \otimes b_k$. A convolution invertible map $\gamma : A \to B$ is such that $\gamma * \gamma^{-1} = \gamma^{-1} * \gamma = \eta_B \circ \varepsilon$, where $\gamma^{-1} : A \to B$ is a linear map, η_B the unit in B viewed as a map and ε the counit in A. We can extend the convolution product to the product of maps $\gamma : A \to B$ and $\sigma : V \to B$ given as a map $\gamma * \sigma : V \to B$ by $\gamma * \sigma = m(\gamma * \sigma)\Delta_{\mathrm{L}}$, where Δ_{L} is the left coaction of A on V. Then we have $\gamma * \sigma(v) = \sum_k \gamma(a_k)\sigma(v_k)$, since $\Delta_{\mathrm{L}}(v) = \sum_k a_k \otimes v_k$. Similarly, we can define $\beta * \sigma : V \to \Omega^1(B)$ by $\beta * \sigma = m'(\beta \otimes \sigma)\Delta_{\mathrm{L}}$, where m' is the product of $\Gamma = \Omega^1(B)$ with B. Finally, we define the convolution product of the maps $\omega_p : A \to \Omega^p(B)$ and $\omega_q : A \to \Omega^q(B)$ by $\omega_p * \omega_q = m''(\omega_q \otimes \omega_q)\Delta$, where m'' is the product in the differential algebra $\Omega(B)$.

Now we are in a position to introduce the notions of quantum gauge, ghost and matter fields and their BRST transformations corresponding to the fields and their BRST transformations occurring in usual quantized gauge theories. To this end, we consider the quantum bundles $P(B, A)$ and $E(B, V, A)$, where the base quantum space B is provided with the graded differential algebra $(\Omega(B), \tilde{d})$ built over $(\Gamma^1, \tilde{d})$ as we have introduced above. Let β be a connection on $P(B, A)$, then it can be put in the form

$$\beta = A_1^0 + c_0^1, \tag{20}$$

where $A_1^0 : A \to \Gamma^{(1,0)}$ and $c_0^1 : A \to \Gamma^{(0,1)}$ are the maps related to the fact that $\beta(a) \in \Gamma^1 = \Gamma^{(1,0)} \oplus \Gamma^{(0,1)}$ for any $a \in B$. This permits us to interpret A_1^0 as the quantum gauge field and c_0^1 as the associated quantum ghost field, where the lower index denotes the degree of the generalized quantum form and the upper one its ghost number. However, a section $\sigma : V \to B$ of $E(B, V, A)$ may be interpreted as the quantum matter field ψ. Furthermore, the curvature $F : A \to \Omega^2(B)$ associated to β can also split into three parts corresponding to the decomposition $\Omega^2(B) = \Omega^{(2,0)}(B) \oplus \Omega^{(1,1)}(B) \oplus \Omega^{(0,2)}(B)$ given by (10), we have

$$F = F_2^0 + F_1^1 + F_0^2. \tag{21}$$

Inserting (20) and (21) into the structure equation

$$F = \tilde{d}\beta + \beta * \beta, \tag{22}$$

and collecting the terms in quantum form degree and ghost number, we obtain

$$F_2^0 = dA_1^0 + A_1^0 * A_1^0, \tag{23}$$

$$F_1^1 = dc_0^1 + QA_1^0 + A_1^0 * c_0^1 + c_0^1 * A_1^0, \tag{24}$$

$$F_0^2 = Qc_0^1 + c_0^1 * c_0^1. \tag{25}$$

Similarly, the Bianchi identity

$$\tilde{d}F + \beta * F - F * \beta = 0 \tag{26}$$

gives

$$dF_2^0 + A_1^0 * F_2^0 - F_2^0 * A_1^0 = 0, \tag{27}$$

$$dF_1^1 + QF_2^0 + A_1^0 * F_1^1 - F_1^1 * A_1^0 + c_0^1 * F_2^0 - F_2^0 * c_0^1 = 0, \tag{28}$$

$$dF_0^2 + QF_1^1 + A_1^0 * F_0^2 - F_0^2 * A_1^0 + c_0^1 * F_1^1 - F_1^1 * c_0^1 = 0, \tag{29}$$

$$QF_0^2 + c_0^1 * F_0^2 - F_0^2 * c_0^1 = 0. \tag{30}$$

Now, in order to determine the quantum BRST transformations of the quantum fields A_1^0 and c_0^1, we remark that the curvature F has more components than required in analogy with the field content of usual gauge theories. Therefore, we impose the following constraints:

$$F_1^1 = 0, \qquad F_0^2 = 0. \tag{31}$$

Notice that these constraints are consistent with the Bianchi identity. The curvature becomes $F = F_2^0$ which may be interpreted as the quantum field strength. It is given by (23) and satisfies (27).

Inserting (31) into (24), (25) and (30), we obtain

$$QA_1^0 = -dc_0^1 - A_1^0 * c_0^1 - c_0^1 * A_1^0, \qquad Qc_0^1 = -c_0^1 * c_0^1,$$
$$QF_2^0 = -c_0^1 * F_2^0 + F_2^0 * c_0^1. \tag{32}$$

The nilpotency of the quantum BRST operator Q and its anticommuting with the differential d are automatically implemented (Eqs. (18) and (19)).

Notice that these quantum BRST transformations simplify further if we consider A as a matrix quantum group. In this case A is generated by noncommuting matrix entries T_b^a, where the matrix R controlling the noncommutativity of the T_b^a obeys the quantum Yang–Baxter equation. Indeed, acting the quantum fields on T_b^a, we write $A_1^0(T_b^a) = A_b^a$, $c_0^1(T_b^a) = c_b^a$ and $F_2^0(T_b^a) = F_b^a$ and the quantum BRST transformations (32) become

$$QA_b^a = -dc_b^a - A_d^a c_b^d - c_d^a A_b^d, \quad Qc_b^a = -c_d^a c_b^d, \qquad QF_b^a = -c_d^a F_b^d + F_d^a c_b^d, \tag{33}$$

since $\Delta(T_b^a) = T_d^a \otimes T_b^d$.

Next, we have also to determine the quantum BRST transformation of the quantum matter field ψ. Here, we use the same procedure as above. First, we introduce the map

$$\varphi : V \to \Gamma^1 \tag{34}$$

defined by $\varphi = \tilde{d}\psi + \beta * \psi$. We decompose then φ as $\varphi = \varphi_1^0 + \varphi_0^1$. So, we obtain

$$\varphi_1^0 = d\psi + A_1^0 * \psi, \tag{35}$$

$$\varphi_0^1 = Q\psi + c_0^1 * \psi. \tag{36}$$

Eq. (35) gives the covariant derivative of the quantum matter field. While Eq. (36) by imposing the constraint $\varphi_0^1 = 0$ gives

$$Q\psi = -c_0^1 * \psi. \tag{37}$$

If we consider A as matrix quantum group and V as a quantum plane generated by T^a_b and v^a, respectively, the quantum BRST transformation (37) simplifies further and becomes

$$Q\psi^a = -c^a_b\psi^b, \tag{38}$$

where $\psi^a = \psi(v^a)$ and the left coaction of A on V is given by $\Delta_{\rm L}(v^a) = T^a_b \otimes v^b$.

To summarise, starting from a quantum principal bundle with connection and its quantum associated vector bundle as introduced in [3], we have constructed the quantum BRST algebra (Eqs. (32) and (37)). At this point, let us recall that in [9] the quantum BRST algebra has been realized in the context of the bicovariant differential calculus by using a classical space–time. In our treatment, we have considered a quantum space–time represented by the base quantum space. In particular, the quantum BRST operator has been introduced through a graded differential algebra (Eqs. (1)–(19)), and the quantum gauge field and its corresponding quantum ghost have been described by the connection (Eq. (20)).

Finally, let us note that the constraints imposed on the curvature (Eq. (31)) correspond to the same fact in the context of the superspace formalism of usual gauge theories [2]. However, in [5] the fields occurring in the topological Yang–Mills theory and their BRST transformations are obtained in terms of an unconstrained superspace formalism (see also [7]). This means that, contrary to what is done in usual Yang–Mills theory, all superfield components of the supercurvature are not constrained. These lead to the introduction of the superpartner of the gauge field and its associated ghost for ghost. The formalism described here and without imposing the constraints (31) permits us also to realize the quantum analogue of the fields in the topological Yang–Mills theory and their BRST transformations. Indeed, we can interpret F^1_1 as the quantum superpartner of the quantum gauge field A^0_1 and F^2_0 as the quantum ghost for ghost of F^1_1. Then Eqs. (24), (25) and (28)–(30) determine the quantum BRST transformations of the fields $(A^0_1, c^1_0, F^0_2, F^1_1, F^2_0)$.

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